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Determinacy of States and Independence of Operator Algebras

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The aim of this paper is to summarize, deepen, and comment upon recent results concerning states on operator algebras and their extensions. The first part is focused on the relationship between pure states and singly generated subalgebras. Among others we show that every pure state ρ on a separable algebra A is uniquely determined by some element of A which exposes ρ.
The main part of this paper is the second section, dealing with characterization of various types of independence conditions arising in the axiomatics of quantum field theory. These two topics, seemingly different, are connected by a common extension technique based on determinacy of pure states.

1. DETERMINACY OF PURE STATES

In this part we will be mainly concerned with Jordan operator algebras. Let us recall that a JB algebra A is a real Banach algebra equipped with a product $\circ$ satisfying the following conditions for all $x, y \in A$:

(i) $x \circ y = y \circ x$
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
(iii) $\|x^2\| = \|x\|^2$
(iv) $\|x^2\| \leq \|x^2 + y^2\|$.

For all details on JB algebras not otherwise discussed here we refer the reader to Hanche-Olsen and Stormer (1984).

An important example of a JB algebra is a self-adjoint part A_{sa} of a C^*-algebra A equipped with a product $x \circ y = 1/2(xy + yx)$. Unlike C^*-algebras, not all JB algebras can be represented as algebras of operators acting on some Hilbert space. Even if this is the case, the resulting algebras need not

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be closed with respect to the usual product of operators. This makes some questions considerably more difficult than in the setting of C^*-algebras.

By a state ρ of a JB algebra we mean a positive and normalized functional. A pure state is defined as an extreme point of the set of all states of the algebra in question.

The restriction and extension properties of pure states with respect to associative algebras have been widely studied in the C^*-algebraic framework (see, e.g., Anderson, 1979; Akemann, 1989). This line of research was originated by Aarnes and Kadison (1969), Kadison (1957) and Kadison and Singer (1950) by showing that any pure state of a separable unital C^*-algebra is restricted to a pure state on some maximal abelian subalgebra. This result was then improved considerably by Akemann (1970), who showed that given finitely many mutually orthogonal pure states $\rho_1, \ldots, \rho_n$ of a separable (not necessarily unital) C^*-algebra A, we can always find a maximal abelian subalgebra B of A such that all restrictions $\rho_i|B$ are pure states of B with unique extension A. For instance, this means that any pure state can be recovered from its restriction to some maximal abelian subalgebra. A natural question arises whether a maximal abelian algebra can be replaced by the smallest possible nontrivial abelian algebra, i.e., by a singly generated algebra. This leads us to the following definition.

Let ρ be a pure state of a JB algebra A. We say that an element $a \in A$ with $\|a\| = 1$ is *determining* for ρ if ρ is the only pure state of A such that $\rho(a) = 1$.

It should be remarked that if a is a determining element of ρ, then the algebra B generated by a is determining in the following sense: the restriction $\rho|B$ is a pure state and extends uniquely to ρ. In this case we call B a *determining subalgebra* for ρ. It follows easily that if B is a determining subalgebra for a given pure state, then the same is true of any subalgebra containing B. Therefore the property of having a determining element is stronger than the property of having a maximal abelian determining algebra.

We show a few important characterizations of states admitting determining elements (Hamhalter, n.d.-a). First of all let ρ be a pure state of an algebra $C_0^R(X)$ of all real-valued continuous functions defined on a locally compact Hausdorff space X vanishing at infinity. (Any associative algebra is isomorphic to some algebra of this type.) Then ρ is represented by a point measure δ_x, $x \in X$. It can be shown that ρ has a determining element if and only if x admits a countable system of neighborhoods (U_n) with intersection $\cap U_n = \{x\}$. As a second example let us consider now the self-adjoint part $A = B(H)_{sa}$ of a C^*-algebra $B(H)$ of all operators acting on some Hilbert space H. It is easy to see that any vector state $\omega_x(a) = (ax, x)$ $(a \in A)$, where $x \in H$, $\|x\| = 1$, has a one-dimensional projection onto span of x as a determining element. Interestingly enough, it can be proved that any state of A admitting a determining element is a vector state.

Our main result is the following criterion of determinacy of pure states.

Theorem 1.1 (see Hamhalter, n.d.-a). Let ρ be a pure state of a JB algebra A. If the left kernel $L_\rho = \{a \in A | \rho(a^2) = 0\}$ has a strictly positive element, then ρ admits a determining element.

The converse implication is true provided that A is unital.

Recall that an element of a JB algebra is strictly positive if every state attains positive value at it. The existence of a strictly positive element is equivalent to the existence of countable approximate unit (σ-unitality). It should be remarked that the assumption of unitality cannot be relaxed from Theorem 1.1.

It is a folklore result that every separable JB algebra admits a strictly positive element. When we specify Theorem 1.1 to a separable case we therefore get that every pure state on a separable JB algebra has a determining element. In other words, a pure state on a separable algebra is always uniquely determined by its restriction to some singly generated subalgebra.

We can also pose the following question: Given finitely many orthogonal pure states, can we find a common, finitely generated associative subalgebra determining for all of them? The answer for the separable case is given in the following theorem. (Let us recall that states ρ, φ are orthogonal if $\|\rho - \varphi\| = 2$.)

Theorem 1.2. Let $\varphi_1, \ldots, \varphi_n$ be pairwise orthogonal pure states on a separable JB algebra A. Then there is a finitely generated associative subalgebra determining for all states $\varphi_1, \ldots, \varphi_n$.

The notion of determining element is useful for the problem of simultaneous extension. As an illustration we shall consider the following situation: We are given a collection (A_α) of subalgebras of a JB algebra A with prescribed pure states φ_α on each A_α and we seek a common pure state extension of states φ_α. Moreover, let us suppose that all states φ_α have a determining element c_α. Then any state φ of A with the property $\varphi(c_\alpha) = 1$ for all α has to be the desired extension. Generally speaking, the presence of a determining element enables us to control given state on a given part.

Another application might be found in the axiomatics of quantum mechanics. If a physical system is modeled by a separable JB algebra then for any pure state of the system we can find a simple classical subsystem given only by one observable such that all information is encoded in it (Theorem 1.1). This contributes also to the discussion on hidden variables in quantum theory.

2. STATISTICAL INDEPENDENCE OF C^*-ALGEBRAS

The aim of this part is to provide a lucid characterization of the statistical independence of C^*-algebras and to establish hitherto unknown relations with other independence conditions of algebraic quantum field theory.

Through this part all C^*-algebras considered are unital with a unit 1. Further, all inclusions of C^*-algebras $B \subset A$ considered have the same unit.

Our central notion is the following. Let A_1 and A_2 be C^*-subalgebras of a C^*-algebra A. We say that a pair (A_1, A_2) is *statistically independent* if for every state φ_1 of A_1 and for every state φ_2 of A_2 there is a state φ of A extending both φ_1 and φ_2.

It is commonly assumed in the mathematical foundations of quantum theory that the system of observables is formed by an operator algebra, while the ensemble of real states of the system is given by its state space. The statistical independence then naturally embodies independence of the corresponding physical systems. Any preparation (i.e., state) of one system cannot effect any preparation (i.e., state) of another system. For that reason the notion of statistical independence was first introduced and studied by Haag and Kastler (1964) in the context of algebraic quantum field theory. According to their independence axiom, algebras corresponding to spacelike-separated regions should be totally uncoupled, i.e., statistically independent. The independence condition is one of the most important parts of quantum field theory and has been widely studied in various forms.

Besides statistical independence we shall deal also with independence in the sense of Schlieder.

A pair (A_1, A_2) of C^*-algebras in a C^*-algebra A is called *S-independent* if $ab \neq 0$ whenever $a \in A_1$ and $b \in A_2$ are nonzero elements.

The S-independence condition describes the position of corresponding algebras without referring to their state spaces. Surprisingly enough, it has been proved by Roose (1969) that a commuting pair of C^*-algebras is statistically independent if and only if it is S-independent.

Finally, we shall deal with logical independence, which is a quantum logic version of the independence condition. This type of independence has been introduced and studied by Redei (1995a, b) in the case of von Neumann algebras. We extend this definition to a more general class of algebras of real rank zero (i.e., to algebras with zero noncommutative Hausdorff dimension). These algebras have been characterized as C^*-algebras for which every self-adjoint element can be approximated by elements with finite spectrum (Brown and Pedersen, 1991). Therefore real-rank-zero algebras have many projections. Many important C^*-algebras have real rank zero (AW^*-algebras, AF-algebras, rotations algebras, Cuntz algebras, Bunce–Deddens algebras, etc).

Let (A_1, A_2) be a pair of algebras of real rank zero contained in an algebra A of a real rank zero. We say that a pair (A_1, A_2) is *logically independent* if for all nonzero projections $p \in A_1$ and $q \in A_2$ there is a nonzero projection $r \in A$ such that $r \leq p$ and $r \leq q$.

In physical formulation, logical independence means that no proposition about one system should imply or be implied by any proposition of the other system. [For a more detailed discussion we refer to Redei (1995a).]

We are now prepared to state results. The key one is the following theorem, the proof of which uses extension technique outlined in the first part of this paper.

Theorem 2.1. A pair (A_1, A_2) of C^*-subalgebras of a C^*-algebra A is statistically independent if and only if for all positive elements $a \in A_1$ and $b \in A_2$ with norm one there is a state φ of A such that $\varphi(a) = \varphi(b) = 1$.

In other words, statistical independence is equivalent to the condition that any couple of independently chosen normalized elements of the corresponding local algebras can be exposed by a common state of the global algebra. This simplifies considerably the original definition. Indeed, going to the universal representation, we see that (A_1, A_2) is statistically independent exactly when for any positive normalized elements $a \in A_1$ and $b \in A_2$ there is a common eigenvector corresponding to eigenvalue one.

Let us now turn to the position of the independence conditions. First of all it can be proved that any statistically independent pair also has to be S-independent (Hamhalter, 1997b). As a consequence of it we get immediately a generalization of a result of Summers (1990) stating that W*-independence, i.e., statistical independence in the category of von Neumann algebras, implies statistical independence.

Redei (1995a, b) poses a question about the position of logical and statistical independence. We answer this problem in the following theorem and counterexample. This result has been proved for von Neumann algebras (Hamhalter, 1997b).

Theorem 2.2. Every logically independent pair (A_1, A_2) of real-rank-zero algebras contained in a real-rank-zero algebra A is statistically independent.

Proof. Assume that a pair (A_1, A_2) is logically independent. By Theorem 2.1 it suffices to prove that given positive normalized elements $a \in A_1$ and $b \in A_2$, we can find a state ρ of A with $\rho(a) = \rho(b) = 1$.

Let us fix a natural number $n \geq 2$. There exist self-adjoint elements $a_n \in A_1$, $b_n \in A_2$ with finite spectrum such that

$$\|a - a_n\| \leq \frac{1}{n}, \qquad \|b - b_n\| \leq \frac{1}{n} \tag{1}$$

Let us write $a_n = \Sigma_{j=1}^{k} \lambda_j p_j$, where λ_j are real numbers and p_j are orthogonal projections in A_1. Since $\|a\| = 1$ and $\|a_n\| = \max_{j=1,\ldots,k} |\lambda_j|$, we can by (1) find λ_{j0}, such that

$$\lambda_{j_0} \geq 0 \qquad \text{and} \qquad 1 - \lambda_{j_0} \leq \frac{1}{n} \tag{2}$$

For arbitrary state φ of A with $\varphi(p_{j0}) = 1$ we have

$$\frac{1}{n} \geq |\varphi(a) - \varphi(a_n)| = |\varphi(a) - \lambda_{j_0}|$$

Therefore,

$$\varphi(a) \geq \lambda_{j_0} - \frac{1}{n} \geq 1 - \frac{2}{n}$$

The same reasoning can be applied for $b \in A_2$. Summing up, for each natural number $n \geq 2$ we can find nonzero projections $p_n \in A_1$, $q_n \in A_2$ such that

$$\varphi(a) \geq 1 - \frac{2}{n}, \qquad \varphi(b) \geq 1 - \frac{2}{n} \tag{3}$$

whenever φ is a state of A with $\varphi(p_n) = \varphi(q_n) = 1$. Since p_n and q_n are nonzero projections, there is a nonzero projection $r_n \in A$ with $r_n \leq p_n$ and $r_n \leq q_n$. Let us now take a state φ_n of A such that $\varphi_n(r_n) = 1$. Then of course

$$\varphi_n(a) \geq 1 - \frac{2}{n}, \qquad \varphi_n(b) \geq 1 - \frac{2}{n} \tag{4}$$

Applying now compactness of the state space of A, we can choose a weak* cluster point ρ of the sequence (φ_n). Obviously, $\rho(a) = \rho(b) = 1$, as required. The proof is complete.

Counterexample 2.3 (see Hamhalter, 1997b). A von Neumann algebra $M = l^\infty \otimes M_5(C)$, where $M_5(C)$ is a matrix algebra of all 5×5 complex matrices, contains two-dimensional subalgebras M_1 and M_2 which are statistically independent but not logically independent.

The interrelations of independence conditions considered in this paper can therefore be described by the following chain of proper implications:

logical independence $\Rightarrow$ statistical independence $\Rightarrow$ S-independence

In the concluding part of our discussion we proceed to the case of commuting algebras. In that case it can be seen easily that S-independence implies logical independence since the product of commuting projections is their infimum. Therefore all independence conditions coincide in this case.

It turns out, perhaps surprisingly, that independence of commuting algebras is given by independence of their centers. The following theorem has been proved in Hamhalter (1997b) for von Neumann algebras. We use the symbol $Z(A)$ for the center of a C^*-algebra A.

Theorem 2.4. Let (A_1, A_2) be a pair of mutually commuting C^*-algebras of real rank zero contained in the algebra A of real rank zero. The following conditions are equivalent.

(i) (A_1, A_2) is logically (statistically, S-) independent.
(ii) $(Z(A_1), Z(A_2))$ is logically (statistically, S-) independent.
(iii) The C^*-algebra generated by $Z(A_1)$ and $Z(A_2)$ is isomorphic to the C^*-tensor product $Z(A_1) \otimes Z(A_2)$.
(iv) The pure state space of the von Neumann algebra generated by $Z(A_1)$ and $Z(A_2)$ is homeomorphic to the product of pure state spaces of $Z(A_1)$ and $Z(A_2)$.

Proof. The proof of equivalency of conditions (ii)–(iv) is the same as in Hamhalter (1997b). The implication (i) $\Rightarrow$ (ii) being trivial, we concentrate on implication (ii) $\Rightarrow$ (i).

Assume that the centers $Z(A_1)$ and $Z(A_2)$ are independent. Let us now take nonzero projections $e \in A_1$ and $f \in A_2$. Denote by $c(e)$ and $c(f)$ the central cover of projections $e \in A_1$ and $f \in A_2$ with respect to the enveloping von Neumann algebras A_1^{**} and A_2^{**}, respectively. Using the Kaplansky density theorem and Pedersen (1979, L. 2.6.2), we have

$$c(e) = \bigvee_{u \in U(A_1)} u^* e u$$

where $U(A_1)$ is a unitary group of A_1. Now,

$$c(e)f = \left(\bigvee_{u \in U(A_1)} u^* e u \right) f = \bigvee_{u \in U(A_1)} u^* e u f \neq 0$$

if and only if there is $u_0 \in U(A_1)$ such that $u_0^* e u_0 f \neq 0$. Since $u_0^* e u_0 f = u_0^* e f u_0$, we have that

$$ef \neq 0 \qquad \text{if and only if} \qquad c(e)f \neq 0$$

By symmetry

$$ef \neq 0 \qquad \text{if and only if} \qquad c(e)c(f) \neq 0 \tag{5}$$

According to (iii), the independence of centers means that algebra B generated by them is isomorphic to the C^*-tensor product $Z(A_1) \otimes Z(A_2)$. It can be verified easily that B^{**} is then isomorphic to the W^*-tensor product $Z(A_1)^{**}$

$\otimes Z(A_2)^{**}$. As a consequence, we see that double duals $Z(A_1)^{**}$, $Z(A_2)^{**}$ are also independent in A^{**}.

By virtue of (5), we now obtain immediately that (A_1, A_2) are independent.

Theorem 2.4 provides a global explanation of the classical result due to Murray and von Neumann (1936, Corollary of Theorem III) saying that the pair of commuting von Neumann algebras is S-independent provided that one of algebras is a factor.

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